Joe Dinius
HW9

MATH 527B
May 16, 2009

1. (M,d) and (N,ρ) are metric spaces. Show that $f : M \to N$ is uniformly continuous iff for every Cauchy sequence $x_n \in M$, $f(x_n)$ is Cauchy sequence in N.

   Solution. ($\implies$) Assume f is uniformly continuous and $x, y \in M$ and $\epsilon > 0$. So, $\exists \delta_\epsilon > 0$ ind. of x, y s.t. $d(x,y) < \delta_\epsilon \implies \rho(f(x), f(y)) < \epsilon$. Let x_n be a sequence be a Cauchy sequence in M. So, $\exists N_{\delta_\epsilon}$ s.t. for $n > m > N_{\delta_\epsilon}$, $d(x_n, x_m) < \delta_\epsilon$. Now, by uniform continuity, I can pick an $\epsilon > 0$ and find an N_{δ_ϵ} s.t. $d(x_n, x_m) < \delta_\epsilon \implies \rho(f(x_n), f(x_m)) < \epsilon$. So now, we have for $\epsilon > 0$ we have $\exists N_{\delta_\epsilon}$ s.t. $n > m > N_{\delta_\epsilon}$, $\rho(f(x_n), f(x_m)) < \epsilon$. Therefore, $f(x_n)$ is Cauchy.

   ($\Longleftarrow$) Assume for every Cauchy sequence $x_n \in M$, $f(x_n) \in N$ is Cauchy. So, given an $\epsilon_1 > 0$, $\exists N_{\epsilon_1}$ s.t. $n > m > N_{\epsilon_1}$, $d(x_n, x_m) < \epsilon_1$. Also, given an $\epsilon_2 > 0$, $\exists N_{\epsilon_2}$ s.t. $n > m > N_{\epsilon_2}$, $\rho(f(x_n), f(x_m)) < \epsilon_2$. Now, I can place restrictions on $N_{\epsilon_1}, \epsilon_1$ s.t. $d(x_n, x_m) < \epsilon_1 \implies \rho(f(x_n), f(x_m)) < \epsilon$, for some $\epsilon > 0$. So, $\exists \delta_\epsilon > 0$ ind. of x, y s.t. $d(x,y) < \delta_\epsilon \implies \rho(f(x), f(y)) < \epsilon$. Therefore, f is uniformly continuous.

   $\therefore$ $f : M \to N$ is uniformly continuous iff for every Cauchy sequence $x_n \in M$, $f(x_n)$ is Cauchy sequence in N.

2. $f : [0,1] \to \mathbb{R}$ is continuous and $f(1) = 1$, $g : [1,\infty) \to \mathbb{R}$ is uniformly continuous and $g(1) = 1$. Define function $h : [0,\infty) \to \mathbb{R}$ by

   $$h(x) = \begin{cases} f(x) & x < 1 \\ 1 & x = 1 \\ g(x) & x > 1 \end{cases}$$

   Show that h is uniformly continuous.

   Solution. Since f is continuous and $[0,1]$ is a compact metric space, and $\mathbb{R}$ is a metric space, by Proposition 1.4.5 f is uniformly continuous (on $[0,1]$). Since f is uniformly continuous on $[0,1]$, there exists $\delta_1 > 0$ such that $|x-y| < \delta_1 \implies |f(x) - f(y)| < \epsilon$ for all $x, y \in [0,1]$. Since g is uniformly continuous on $[1,\infty)$, there exists $\delta_2 > 0$ such that $|x-y| < \delta_2 \implies |f(x) - f(y)| < \epsilon$ for all $x, y \in [1,\infty)$. So let $\delta = \min\{\delta_1, \delta_2\}$. Then clearly if both x and y are in $[0,1]$ or are both in $[1,\infty)$, the uniform continuity condition for h is satisfied. So fix $x \in [0,1]$ and $y \in [1,\infty)$, and assume $|x-y| < \delta$. Then we have that $|x-1| < \delta$ and $|1-y| < \delta$ also, so that $|f(x) - f(1)| < \epsilon/2$ and $|g(1) - g(y)| < \epsilon/2$ by the uniform continuity of f and g. Thus,

   $$\begin{aligned} |h(x) - h(y)| &\le |h(x) - h(1)| + |h(1) - h(y)| \\ &= |f(x) - f(1)| + |g(1) - g(y)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

   So h is uniformly continuous on $[0,\infty)$.

3. $f : [0, \infty) \to \mathbb{R}$ is bounded and continuous.

   (a) Does it follow that f is uniformly continuous? Prove or give a counterexample.

   Proof. No, it doesn't follow that f is uniformly continuous. Consider the continuous, bounded function $f(x) = \sin(x^2)$. Now, consider the sequences $x_n = \sqrt{2n\pi}$ and $y_n = \sqrt{(2n + 1/2)\pi}$. Clearly, as $n \to \infty$, $|x_n - y_n| \to 0$, however $|f(x_n) - f(y_n)| = 1$. Therefore, let $\epsilon = 3/4 < 1$. Therefore, $\nexists \delta > 0$ s.t. $|x_n - y_n| < \delta \implies |f(x_n) - f(y_n)| < 3/4$. So, f is not uniformly continuous.

   (b) If in addition we are given that $\lim_{x\to\infty} f(x)$ exists, show that f is uniformly continuous.

   Solution. Pick some $\epsilon > 0$. Since $\lim_{x\to\infty} f(x)$ exists, $\exists N(\epsilon)$ s.t. for all $x, y \geq N(\epsilon)$, $|f(x) - f(y)| < \epsilon$. We know that the interval $[0, N(\epsilon) + 1]$ is compact, and therefore f is uniformly continuous on this interval. That is, $\exists \delta(\epsilon) > 0$ s.t. $|x - y| < \delta(\epsilon)$ and $x, y \in [0, N(\epsilon) + 1] \implies |f(x) - f(y)| < \epsilon$. Let $\delta = \min(\delta(\epsilon), 1)$. Now, $|x - y| < \delta$ implies that either (1) $x, y \in [0, N(\epsilon) + 1], |x - y| < \delta(\epsilon)$ or (2) $x, y > N(\epsilon)$. In both (1) and (2), $|f(x) - f(y)| < \epsilon$ follows directly.

   $\therefore$ f is uniformly continuous.

4. Problems 1.4.13 and 1.4.14. Pg. II-317.

   (1.4.13) Let f be continuous on $[0, 1]$. The *modulus of continuity* of f is defined by

$$\omega(f; \delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|.$$

   Prove that $\lim_{\delta\to 0} \omega(f; \delta) = 0$.

   Proof. In order to show that the limit converges to 0, we need that given some $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ s.t.

$$\omega(f; \delta) < \epsilon \quad \forall \delta < \delta(\epsilon).$$

   Pick some $\epsilon > 0$. We know that f is uniformly continuous; since f is continuous on a compact set. Therefore, $\exists \eta(\epsilon)$ s.t.

$$|x - y| < \eta(\epsilon) \implies |f(x) - f(y)| < \epsilon.$$

   So, for any $\delta < \eta(\epsilon)$,

$$\begin{aligned} \omega(f; \delta) &= \sup_{|x-y|\leq\delta} |f(x) - f(y)| \\ &\leq \sup_{|x-y|<\eta(\epsilon)} |f(x) - f(y)| \\ &< \epsilon. \end{aligned}$$

   Now, we can pick $\delta(\epsilon) = \eta(\epsilon)$ and conclude that

$$\lim_{\delta\to 0} \omega(f; \delta) = 0.$$

(1.4.14) Let (M, d) and (M', d') be metric spaces, and suppose that $f : M \to M'$ is uniformly continuous. Prove: if $d(x_n, y_n) \to 0$, then $d'(f(x_n), f(y_n)) \to 0$. Is this true if f is not uniformly continuous?

Proof. Assume f is uniformly continuous. Therefore, there exists a $\delta(\epsilon)$ independent of x, y s.t. $d(x, y) < \delta(\epsilon) \implies d(f(x), f(y)) < \epsilon$. Suppose now that $d(x_n, y_n) \to 0$. Therefore, there exists an index $N_{\delta(\epsilon)}$ s.t. for all $n > N_{\delta(\epsilon)}$, $d(x_n, y_n) < \delta(\epsilon)$. So, now for all $n > N_{\delta(\epsilon)} = N_\epsilon$,

$$d(x_n, y_n) < \delta(\epsilon) \implies d'(f(x_n), f(y_n)) < \epsilon.$$

So, we have that $\exists N_\epsilon$ for any given $\epsilon > 0$ s.t. $n > N_\epsilon \implies d'(x_n, y_n) < \epsilon$.

$\therefore$ if $d(x_n, y_n) \to 0$, then $d'(f(x_n), f(y_n)) \to 0$.

♣

If f is not uniformly continuous, then the statement just proven is no longer true. For example, consider the metric space $(M, |\cdot|)$, where $M = (0, 1)$, $f(x) = 1/x$ and the sequences $x_n = 1/n, y_n = 1/(n + 1)$. Now,

$$\begin{aligned} |x_n - y_n| &= \left|\frac{1}{n} - \frac{1}{n+1}\right| \\ &= \frac{1}{n(n+1)} \\ &\to 0. \end{aligned}$$

But, now consider

$$\begin{aligned} |f(x_n) - f(y_n)| &= |n - (n + 1)| \\ &= 1 \ \ \forall n. \end{aligned}$$

Therefore, $|f(x_n) - f(y_n)| = 1 \not\to 0$ even despite the fact that $|x_n - y_n| \to 0$.

5. Problems 1.4.41 and 1.4.42. Pg. II-332.

(1.4.41) Find the pointwise limit of

$$f_n(x) = \frac{nx^2 - 1}{nx^2 + 1}$$

as $n \to \infty$. Is the convergence uniform?

Solution. By L'Hospitâl, $f_n(x) \to 1$ for $x \neq 0$, otherwise $f_n(0) = -1$. For $x \neq 0$,

$$\begin{aligned} |f_n(x) - 1| &= \left|\frac{nx^2 - 1}{nx^2 + 1} - 1\right| \\ &= \left|\frac{nx^2 - 1}{nx^2 + 1} - \frac{nx^2 + 1}{nx^2 + 1}\right| \\ &= \left|-\frac{2}{nx^2 + 1}\right| = \frac{2}{nx^2 + 1}. \end{aligned}$$

Clearly, given some $\epsilon > 0$, $\exists N$ s.t. $n > N \implies |f_n(x) - 1| < \epsilon$. But, from the relation above, this N depends on x. Therefore the convergence is *NOT* uniform.

(1.4.42) Find the pointwise limit of

$$f_n(x) = \left(\frac{nx^2 - 1}{nx^2 + 1}\right)^2$$

as $n \to \infty$. Is the convergence uniform? Find $\lim_{n\to\infty} f_n'(x)$. Comment on the relation to propositions in this subsection.

Solution. For this exercise, unlike (1.4.42), $f_n(x) \to 1\ \forall x$. Following the same steps as in (1.4.42), we get

$$\begin{aligned} |f_n(x) - 1| &= \left|\frac{n^2x^2 - 2nx^2 + 1}{(nx^2+1)^2} - \frac{n^2x^2 + 2nx^2 + 1}{(nx^2+1)^2}\right| \\ &= \frac{4nx^2}{(nx^2+1)^2} \end{aligned}$$

Consider $x = \sqrt{1/n}$. Then the above limit, for all n, is $|f(\sqrt{1/n}) - 1| = 1$. Clearly, the convergence is not uniform, since the difference cannot be made arbitrarily small through selection of an N.

Now, for the derivative, applying differentiation rules yields

$$f_n'(x) = \frac{8n^2x^3 - 8nx^2}{(nx^2+1)^2}.$$

Clearly, by L'Hospitâl's rule, this derivative tends to 0 $\forall x$. This makes sense intuitively, $f_n(x) \to 1$. The propositions from Flaschka don't apply here, since they assume that both f_n and f_n' converge uniformly. We just happened to get lucky that the corrective derivative limit was reached.

6. Problems 1.4.45 and 1.4.48. Pg. II-333.

   (1.4.45) What is the negation of "$\{f_n\}$ converges to uniformly on $[0, 1]$"? (I.e., how would you start a proof by contradiction?)

   Solution. Let $f(x)$ be the pointwise limit for all x. First, a reiteration of the original statement (to converge uniformly): given $\epsilon > 0$, $\exists N(\epsilon)$ s.t. $n > N(\epsilon)$ and $x \in [0, 1]$, $|f_n(x) - f(x)| < \epsilon$.

   The negation of this statement is: $\exists \epsilon > 0$ s.t. $\forall N$ there are constants $n \geq N$ and $x \in [0, 1]$, $|f_n(x) - f(x)| \geq \epsilon$. In this construction, both the n and x depend on choice of N.

   (1.4.48) What is the negation of "$\int_0^\infty f(t, x)dx$ converges uniformly for $t \in [-1, 1]$"?

   Solution. Let $g(t)$ be the pointwise limit for all $t \in [-1, 1]$. To converge uniformly means: given an $\epsilon > 0$, $\exists M$ s.t. for all $t \in [-1, 1]$, $|\int_0^M f(t, x)dx - g(t)| < \epsilon$.

   The negation of this statement is: $\exists \epsilon > 0$ s.t. $\forall M > 0$ there is a $t \in [-1, 1]$ where $|\int_0^M f(t, x)dx - g(t)| \geq \epsilon$. In the negation, t depends upon choice of M.

7. (1.4.49) Pretend that you do not know the function defined by e^x defined by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Prove that this series can be differentiated and integrated, term by term, as is often desired.

Proof. We need to show that the sum converges uniformly. Define $f_n(x) = x^n/n!$, and let $R > 0$ be a radius of uniform convergence for the sum. By the Weierstrauss M-test, for large enough n the series becomes bounded by a geometric series $|f_n| \leq R^n/n!$. So, the series converges uniformly, and can be differentiated once to yield

$$\begin{aligned} \frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^n}{n!} &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

where equality follows from re-indexing the sum. Since the series differentiates to the original series, differentiation term-by-term can be performed as many times as desired.

Integration as often as desired follows directly from the fact that differentiation can be performed as often as desired. The long-winded explanation involves re-indexing the sum the same as before after term-by-term integration.

(1.4.50) True or false:

$$\int_0^1 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} dx = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}?$$

Solution. This statement is true. It can be shown that both of the sums above converge uniformly. Considering just the LHS, we have that if $\cos nx/n^2 \to g(x)$ pointwise

$$\begin{aligned} \left|\sum_{n=1}^{N} \frac{\cos nx}{n^2} - g(x)\right| &= \left|\sum_{n=N+1}^{\infty} \frac{\cos nx}{n^2}\right| \\ &\leq \left|\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right| \end{aligned}$$

which can be made arbitrarily small (independent of x). The same result follows for the RHS. The previous exercise shows that we can perform term-by-term differentiation given uniform convergence. Now consider,

$$\frac{d}{dx}\sum_{n=1}^{\infty} \frac{\sin nx}{n^3} = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Taking the integral from 0 to 1 finishes the problem.

8. $f(x,t)$ is continuous in x and t. Further, the (Riemann) integral $\int_0^\infty f(x,t)dt$ converges uniformly for $x \in [-1,1]$ to a function $F(x)$. Show that F is continuous on $[-1,1]$.

Solution. Given some $\epsilon > 0$, we need to find a $\delta(\epsilon)$ s.t.

$$|x-y| < \delta(\epsilon) \implies |F(x) - F(y)| < \epsilon.$$

By uniform continuity of $\int_0^\infty f(x,t)dt$, we know that $\exists R_\epsilon$ s.t. for all $x \in [-1,1]$,

$$\big|F(x) - \int_0^{R_\epsilon} f(x,t)dt\big| < \frac{\epsilon}{3}$$

Now, by continuity in x of $f(x,t)$ on $[-1,1]$ we have that for a given $\epsilon > 0$, $\exists\delta(\epsilon) > 0$ s.t. $|x-y| < \delta(\epsilon) \implies |f(x,t) - f(y,t)| < \epsilon/(3R_\epsilon)$. From this we can conclude

$$\begin{aligned}\big|\int_0^{R_\epsilon} f(x,t)dt - \int_0^{R_\epsilon} f(y,t)dt\big| &\leq \int_0^{R_\epsilon} |f(x,t) - f(y,t)|dt \\ &< \int_0^{R_\epsilon} \frac{\epsilon}{3R_\epsilon} dt \\ &= R_\epsilon \frac{\epsilon}{3R_\epsilon} = \frac{\epsilon}{3}.\end{aligned}$$

Putting all of this information together and using the triangle inequality, we have

$$\begin{aligned}|F(x) - F(y)| &\leq \big|F(x) - \int_0^{R_\epsilon} f(x,t)dt\big| + \big|\int_0^{R_\epsilon} f(x,t)dt - \int_0^{R_\epsilon} f(y,t)dt\big| \\ &+ \big|\int_0^{R_\epsilon} f(y,t)dt - F(y)\big| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Uniform convergence allowed us to use the same R_ϵ for both x and y. Therefore, $F(x)$ is continuous.

9. Define

$$f(\omega,t) = \begin{cases} \frac{\sin \omega t}{t} & t \neq 0 \\ \omega & t = 0 \end{cases}$$

(a) Show that f is continuous in t and ω.

Solution. First consider ω. If $t = 0$, $f(\omega, 0) = \omega$ is continuous. If $t \neq 0$, f is differentiable with $f_\omega(\omega,t) = \cos \omega t$, therefore f is everywhere continuous w.r.t ω.

For $t \neq 0$, f is continuous w.r.t to t because it is differentiable. Using L'Hospitâl's rule we can show that f is continuous w.r.t t at $t = 0$:

$$\lim_{t\to 0} f(\omega,t) = \lim_{t \to 0} \omega \cos \omega t = \omega.$$

So, f is continuous w.r.t t.

$\therefore$ f is continuous in t and ω.

Let the Riemann integral define

$$I(\omega) = \int_0^\infty f(\omega, t)dt.$$

(b) Define what it means for the integral to converge uniformly for $\omega \in [-1, 1]$.

Solution. For the integral to converge uniformly means: given any $\epsilon > 0$ $\exists R_\epsilon$ (independent of ω) s.t. $|I(\omega) - \int_0^s f(\omega, t)dt| < \epsilon$ whenever $s > R_\epsilon$ for all $\omega \in [-1, 1]$.

(c) Evaluate $I(\omega)$ for $\omega \in [-1, 1]$. Does the integral converge uniformly?

Solution. We need a contour integral to evaluate $I(\omega)$. Consider

$$\oint_\Gamma \frac{e^{i\omega z}}{z} dz.$$

Take Γ to be the contour of the upper half plane with a small cut (of radius ϵ) around the singularity at $z = 0$. By the Residue thm, we can conclude that

$$\oint_\Gamma \frac{e^{i\omega z}}{z} dz = \int_0^\pi ie^{i\omega Re^{i\theta}} d\theta + \int_{-R}^{\epsilon} \frac{e^{i\omega x}}{x} dx + \int_\pi^0 ie^{i\omega\epsilon e^{i\theta}} d\theta + \int_\epsilon^R \frac{e^{i\omega x}}{x} dx = 0$$

since no poles are enclosed. In a previous assignment last semester, we showed that the first integral on the RHS, $\int_0^\pi ie^{i\omega Re^{i\theta}} d\theta = 0$. We can rearrange the remaining integrals (by changing indices) to get the following:

$$\int_\pi^0 ie^{i\omega\epsilon e^{i\theta}} d\theta + \int_\epsilon^R \frac{e^{i\omega x} - e^{-i\omega x}}{x} dx = 0$$

Take the limit as $\epsilon \longrightarrow 0$,

$$\begin{aligned} \int_\epsilon^R \frac{e^{i\omega x} - e^{-i\omega x}}{x} dx &= -\int_\pi^0 id\theta \\ &= i\pi \end{aligned}$$

and then notice that $\frac{e^{i\omega x} - e^{-i\omega x}}{x} = 2i\frac{\sin \omega x}{x}$. Take the limit as $R \longrightarrow \infty$, we get that

$$\int_0^\infty \frac{\sin \omega x}{x} dx = \operatorname{sgn}(\omega)\frac{1}{2i} i\pi = \operatorname{sgn}(\omega)\frac{\pi}{2}.$$

Following these steps for different signs of ω we get

$$I(\omega) = \begin{cases} \frac{\pi}{2} & \omega > 0 \\ 0 & \omega = 0 \\ -\frac{\pi}{2} & \omega < 0 \end{cases}$$

Since $I(\omega)$ is not continuous, we can conclude that the integrals cannot converge uniformly (from part (b) of this exercise).

(d) Show that there exists a continuous function $g : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ s.t. the Riemann integral defines

$$H(\omega) = \int_0^\infty g(\omega, t) dt$$

for all $\omega \in [-1, 1]$ but $H(\omega)$ has a dense set of discontinuities.

Solution. Let $\{r_n\}_{n=1}^\infty$ be an enumeration of the rational numbers on $[-1, 1]$. This set is dense in $[-1, 1]$. Define

$$f_n(\omega, t) = \begin{cases} \frac{\sin t(\omega - r_n)}{2^n t} & t \neq 0 \\ \frac{\omega - r_n}{2^n} & t = 0. \end{cases}$$

From part (c), we know that $\int_0^\infty f_n dt$ is discontinuous at $\omega = r_n$. Let

$$g(\omega, t) = \sum_{n=1}^\infty f_n(\omega, t).$$

This sum converges uniformly since $\sup_{t,\omega} f_n \leq 2^{-n+1}$, which is convergent in the sum. The uniform convergence of the sum also provides us with continuity of g in both ω and t. So,

$$H(\omega) = \int_0^\infty \sum_{n=1}^\infty f_n(\omega, t) dt = \sum_{n=1}^\infty \int_0^\infty f_n(\omega, t) dt.$$

Therefore H has a dense set of discontinuities.